HPM 2024
ABSTRACTS

ORAL PRESENTATIONS
WORKSHOPS, AND POSTERS♦

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ORAL PRESENTATIONS

Marlene Kafui Amusuglo* and Antonín Jančařík

Exploring the Interplay of Culture and History in Ghanaian Mathematics.

This research delves into the complex interplay of culture and history in the Ghanaian early grade mathematics curriculum, offering insights into how cultural and historical contexts shape the teaching and learning of mathematics. In Ghana, where cultural diversity is celebrated and historical legacies run deep, integrating culture and history into the mathematics curriculum holds significant potential for fostering meaningful connections and promoting inclusive education.

The curriculum design reflects Ghana's commitment to contextualized education, drawing upon indigenous knowledge systems, historical narratives, and cultural practices to enrich the learning experience. Educators strive to create a learning environment that resonates with students' experiences and cultural identities by incorporating culturally relevant examples, local contexts, and traditional problem-solving methods. By grounding mathematical concepts in familiar cultural contexts, the curriculum seeks to enhance student's understanding and engagement while affirming the value of their cultural heritage.

Including historical perspectives provides students with a deeper appreciation of the evolution of mathematical ideas and their relevance to contemporary society. From the ancient Ghanaian civilization's mathematical achievements to the impact of colonialism on mathematical education, historical narratives offer valuable insights into the dynamic relationship between mathematics and culture over time. By contextualizing mathematical concepts within historical frameworks, educators aim to cultivate critical thinking skills and encourage students to question and challenge conventional perspectives. A qualitative content analysis will be used to compare the Ghanaian early grade mathematics curriculum. According to Wallen and Fraenkel (2001), content analysis examines document contents, whether textual or visual. In addition, Graneheim et al. (2017) argued that content analysis should serve a meaningful function in research, contributing vital knowledge to the subject of study or generating information beneficial in assessing and improving social or educational activities. It is a technique for objectively extracting the characteristics of the information from a document's content. In conclusion, exploring culture and history in the Ghanaian early grade mathematics curriculum offers valuable insights into the potential of contextualized education to promote student engagement, enhance learning outcomes, and foster cultural pride.

Évelyne Barbin

Signs and diagrams: on visualization in history of mathematics and in teaching.

In *The Mathematical Analysis of Logic* (1847), George Boole emphasized the importance of using symbols with a full understanding of their meaning and he added that, perhaps, the best security against the danger of an unreasoning reliance upon symbols would be to treat them in the spirit of the methods that were known at the time when their application was developed.

We propose such a return to history for the purpose of designating and signifying mathematical signs, symbols or diagrams in four historic key moments: 1) Greek geometry in Euclid's *Elements*; 2) Cartesian method in René Descartes' *Geometry* and Antoine Arnaud's *New Elements of Geometry*; 3) Symbolism in George Boole's *Mathematical Analysis of Logic*; and 4) Graphical methods in Charles-Ange Laisant's *Mathematical Initiation*.

We thus examine different types of visualization in geometry, algebra, and logic and roles in verifying, applying a method, or inventing. For this, we use the classification of signs and the design of diagrams introduced by Charles Sanders Peirce.

Our purpose is to present different uses of signs, schemas, and writing within our teaching of mathematics to students: not only when and how they were introduced in history, but also with what goals, such as to hold a discussion on figures, to give an operating status to objects, to write a generalization, to represent relations between objects, or to transport practices and knowledge from one mathematical field into another. This does not mean ignoring the difficulty of linking "what we see" with "what we say".

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Janet Heine Barnett

Learning Abstract Algebra via Primary Historical Sources: An Existence Proof.

In his 2008 HPM plenary lecture, David Pengelley argued that we should rebuild the entire mathematics curriculum at all levels around translated primary sources studied directly by our students, and offered an example of how one particular topic can be taught via a student project based on primary source material. In this presentation, I report on an Abstract Algebra course that provides an existence proof for the possibility of offering entire courses in the core curriculum using such materials.

I begin with a description of the "textbook" for the course: a series of Primary Source Projects (PSPs) based on original papers by five mathematicians (Lagrange, Cauchy, Cayley, Dedekind, Hölder) of importance in the historical development of abstract algebra. Each PSP employs a selection of excerpts from these original papers, together with secondary commentary that elaborates on the excerpts and a sequence of tasks that are interspersed throughout the project for students to complete as they work through the source material.

I then briefly examine these PSPs through the lens of Sfard's theory of *commognition*, a term that combines "cognition" and "communication" to emphasize that these two human activities are "but different manifestations of basically the same phenomenon" (Sfard 2008, p. 83). Within this framework, "doing" mathematics at either the individual or community level is the act of participating in mathematical discourse, and "learning" that of becoming a full-fledged participant in a discourse community. This view is particularly applicable for critically examining materials aimed at teaching mathematics via primary sources, as it treats mathematics education research (i.e., studying the evolving discourse of an individual) and historical research (i.e., studying the historical evolution of mathematical discourses) as different versions of the same endeavor.

I close with observations about students' responses to the course, which I propose as an existence proof of another phenomenon: the positive consequences that can result by shifting away from an acquisitionist conception of learning in which instructors strive to remove potential impediments to student learning, towards a participationist view in which students and instructors talk and think together in order to overcome communication obstacles that are not only natural, but sometimes necessary.

The development of the curricular materials and the collection of student data described in this presentation were supported in part by funding from the US National Science Foundation.

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Kathleen M. Clark and Janet Heine Barnett*

TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources: Looking to the future, with an eye on the past.

The "TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources" (TRIUMPHS) project was a multi-year, multi-institution effort to design and test student projects based on the reading of primary historical source materials, and to research their impact on students and instructors. When HPM met in Montpellier in 2016, TRIUMPHS was just completing its first year of funding from the US National Science Foundation. When that funding ended in 2023, the original group of seven principal investigators had expanded to include nine doctoral students in mathematics and mathematics education, thirteen additional authors, and a sizeable cadre of site testers. This presentation takes a reflective look back at what TRIUMPHS accomplished (or not) with regard to the development and classroom implementation of its collection of "Primary Source Projects" (PSPs), and a realistic look ahead at the prospects for sustaining this effort into the future.

Some highlights from TRIUMPHS's accomplishments, and questions associated with each, that will be considered in this presentation:

- Development of over 100 PSPs for use in university (tertiary) mathematics courses ranging from beginning algebra to topology.
    - What do PSPs look like? Who writes them, and how often?
    - How have PSPs been reviewed for consistency and quality? What aspects of the review process could be continued in the future?
    - Should new PSP development continue? If so, who will write them? What does it take to learn to write one?
- Classroom testing of PSPs by over 120 university mathematics instructors at a wide variety of institutions across the US and Canada.
    - What does classroom implementation of PSPs look like? Who teaches with them, and how often?
    - Which PSPs get used the most (or least) frequently? Do these PSPs have common features that might explain these trends?
    - How have site testers been recruited and supported? What aspects of site tester support could be continued in the future?

This presentation will be of interest to instructors who wish to bring primary sources into their own teaching, as well as those involved in widespread implementation of instructional innovations and researchers who study the effect of using history to teach mathematics.

Dirk De Bock* and Wendy Goemans

Wiskunde Post, a mathematical magazine for students supporting the modern mathematics movement in Flanders.

After World War II, mathematics magazines for young people appeared in many countries. Some of these magazines focused only on mathematics (e.g., *Pythagoras* in the Netherlands, *Mathematical Pie* in the United Kingdom), others on both mathematics and physics (e.g., *Le Facteur X* in France, *Kvant* in the Soviet Union). To stimulate young people's interest in mathematics, these magazines often presented mathematical puzzles, challenging problems, and elements from the history of mathematics. Some of these journals also played an important role in the spread of "modern mathematics" in the 1960s (e.g.*, Appendix of the Bulletin of HMS* in Greece, *Wiskunde Post* [*Mathematics Post*] in Flanders, the Dutch-speaking part of Belgium). However, with few exceptions, the role of mathematics journals for young people, in particular their role in the modern mathematics movement, has to date been an underexposed topic in the history of mathematics education. In this communication, we examine the ways in which *Wiskunde Post* supported the "modern mathematics" movement in Flanders.

In the 1960s, the introduction of "modern mathematics" in secondary schools was in full preparation in Belgium. The main initiative to prepare Belgian teachers for the arrival of "modern mathematics" in their classrooms were the "Days of Arlon", a series of in-service teacher training courses organized annually from 1959 to 1968 in the very south of (French-speaking) Belgium (De Bock & Vanpaemel, 2019). Flemish mathematics teachers also participated in these courses, though to a lesser extent than their French-speaking colleagues.

In the 1961–1962 school year, *Wiskunde Post*, a "modern" magazine on mathematics was launched in Flanders. It appeared until the 1973–1974 school year, 53 issues in total. *Wiskunde Post* was intended for secondary school students, but it was also eagerly read by Flemish mathematics teachers, many of whom were introduced to "modern mathematics" through this magazine. The journal had three tracks — history of mathematics, "traditional mathematics," and "modern mathematics" — but the latter predominated. Students could learn about the 20th-century mathematics of sets, algebraic structures, linear algebra, topology, and discrete mathematics or, as the editorial team had already announced in the first issue, "we will deal resolutely with the most modern mathematical views."

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Burcu Durmaz* and Hanna Nadim Haydar*

Educators Navigating the Intersection of Elementary Mathematics, Storytelling, Identity, and History: Illustrations from the Islamic Context.

One way to offer engaging activities for students in mathematics lessons is to draw upon examples and principles from the history of mathematics (Agterberg et al., 2022). But it is difficult for mathematics teachers to find and use appropriate historical content for their students (Girit Yıldız & Ulusoy, 2023) especially at the elementary level (Haydar & Durmaz, 2022). So, teachers require explicit professional development regarding the utilization of the history of mathematics in various ways within mathematics classrooms (Agterberg et al., 2022). In addition to this, the use of stories from the history of mathematics can contribute not only to teachers' competences but also to the development of students' mathematical identities through the use of contexts from their own culture. This mathematizing perspective, integrated with culturally responsive pedagogy, can support the transformation of stories drawn from the history of mathematics from window stories (external and distant) to mirror stories (relatable and reflective) (Sims-Bishop, 2011). The aim of this study, which is based on this perspective, is to explore how primary school and middle school mathematics teachers can use stories from the history of mathematics to help their students (grades 2–6) develop positive mathematical identities and mindsets. To this end, using a participatory action research methodology, a rubric was developed to guide the selection of historical mathematical stories; mathematical stories from the historical Islamic context that corresponded to the curriculum objectives were identified; these stories were classified according to grade level, learning domain and story topic (mathematician, event); modules were created by writing stories consisting of multimodal narratives; curriculum materials and teacher guides related to the stories were created; pilot studies were conducted; and, finally, feedback was obtained through questionnaires and focus groups. According to the initial findings, students show a strong interest in lessons that combine mathematics with history and storytelling, highlighting the necessity to expand the representation of mathematicians and tackle mindset misconceptions. Feedback from teachers affirms the significance of historical mathematics narratives in facilitating mathematizing and underscores the effectiveness of incorporating multidisciplinary connections into our curriculum design. As a result, a discussion about how teachers can use stories from the history of mathematics to provide culturally responsive mathematics education to their students is emphasized.

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Celil Ekici

Elementarisation of Mathematics for Undergraduates by Integrating Historical Stances with Trigonometric Functions towards Fourier Methods.

As mathematics advances, a mathematical idea once considered complex can become elementary in the sense that it becomes fundamental for higher mathematics. From the perspective of undergraduate mathematics education, such a mathematical idea can become essential for upper-level courses for applied mathematics and STEM majors. Building on Felix Klein's conception, elementarisation provides a dynamic perspective describing the history of mathematics and its relationship to mathematics education and undergraduate mathematics. The concept of hysteresis, as emphasized by Klein and Schubring, relates to this process of elementarisation through which mathematical ideas become more elementary and foundational based on recent advances and practices with mathematics and its applications. A mathematical idea or practice such as combining trigonometric functions becomes elementary in building and applying mathematics for the further development of mathematics and its applications. New stances on mathematical ideas are gained as these ideas are recontextualized, reclarified, and redefined in the advancement of mathematics and mathematical sciences. Due to the historical restructuring of the applied mathematics found in upper-level courses in STEM disciplines, students are compelled to grasp a higher conception of a mathematical idea or a practice, such as the Fourier methods related to trigonometric functions that arise in courses on topics such as control systems and signal analysis. Hysteresis orients a learner towards building a higher stance on a mathematical concept such as trigonometric functions through an understanding of the progressive restructuring of its elements as experienced in the historical development of mathematics and its applications in STEM disciplines. As mathematics advances while applying its concepts and practices, mathematical practices, such as those related to trigonometric functions, are reclarified through their progressive restructuring which reorients mathematical ideas and facilitates their advancement and applications.

This paper provides an exemplification of the elementarisation process by presenting, discussing and analyzing multiple historical stances on trigonometric functions, including circular, hyperbolic, elliptical, complex, finite, higher dimensional spaces, and geometric algebra. The historical approach behind the Fourier analysis stance with its orientation towards trigonometric functions as bases of function spaces will be emphasized. Adopting the concept of hysteresis as it relates to trigonometric functions, it will be argued that renewed foundational developments on trigonometric functions should enter into the undergraduate mathematics and school mathematics after this process of elementarisation. The implications of delayed hysteresis and neglect will be discussed from the perspectives of engineering mathematics education and physics education. Recommendations for learning trajectories of elementarised mathematics for integration into core undergraduate mathematics courses for STEM majors will be presented. This hysteresis process can inform curricular restructuring by revisiting the question of what mathematical ideas and practices are fundamental for the advancement of mathematics and mathematical sciences. This process thus places the undergraduate mathematics curriculum for STEM disciplines in a productive relation with the progress of mathematics and its fundamental applications in STEM disciplines.

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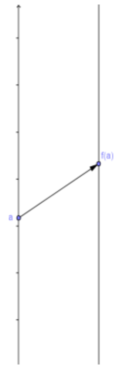
Martin Flashman

Two Examples from History: Mapping Diagrams to Visualize Relations and Functions.

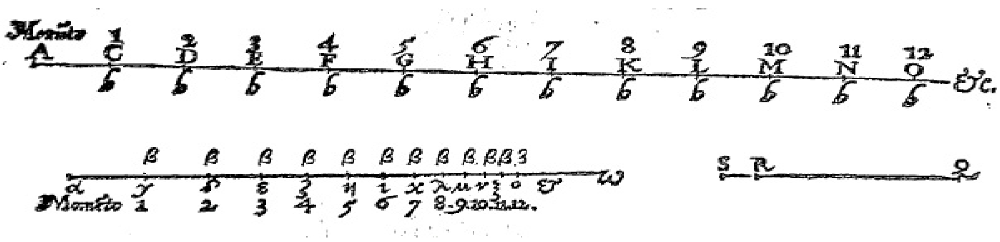


A mapping diagram for a function, *f*, is a figure consisting of two parallel number lines (or axes) and a set of arrows between these lines. Points on one (source or input) line represent numbers from the domain of *f*, the source (controlling or independent) variable values. Points on the other (target or output) line represent numbers from the co-domain of *f*, the target (controlled or dependent) variable values. An arrow in the diagram has its tail on a point on the source (domain), representing a selected number, *a*. The head of the arrow points to the function value, *f* (*a*), for the number *a*, represented by a point on the target (co- domain) line.

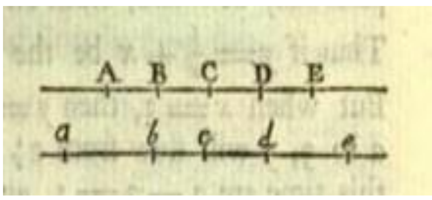
Mapping diagrams are also described as function diagrams, arrow diagrams, dynagraphs, parallel coordinate graphs, or cographs. They visualize functions and relations as an alternative to cartesian graphs.

The history of mapping diagrams predates Descartes' work *La Géométrie* (1637) which introduced numbers and equations to the analysis of geometry. Though originally an appendage to his *Discourse on Method*, this mathematical work eventually became the basis for current coordinate geometry and the graphical representation of functions.

Published in 1614, Napier's work, *Mirifici logarithmorum canonis descriptio* (*A Description of the Wonderful Table of Logarithms*) used mapping diagrams as key visualizations for the introduction to logarithms. His diagrams did not use arrows to indicate the corresponding points. Instead, Napier used alphabetic labels to connect a point moving on one axis with its position increasing arithmetically to a second point moving on a parallel line segment with its position decreasing geometrically with respect to the segment's endpoint.



Later in history mapping diagrams were used by Isaac Newton in his *Treatise of the Method of Fluxions* (published posthumously in 1736) to visualize his solution of the problem of finding the relation of two quantities, given an equation involving their fluxions.



The author will explain and connect these two examples (using GeoGebra) to mapping diagrams in contemporary approaches to logarithms and differential equations, illustrating how this history can be integrated into current pedagogy.

James Franklin

Applied Mathematics First, Pure Second.

Mathematics is taught on a Platonist schema. According to a Platonist philosophy, mathematics is about a world of *abstracta* like numbers, sets and vector spaces, and the pure mathematical results about them are, down the track, "applied" to questions arising in physics, biology, finance and so on. Most students' mathematical education stops before any serious engagement with those areas, so students are often trapped in a world of unmotivated abstractions. That has inevitable negative impact on interest and understanding, while employers complain that mathematics graduates know little about how to use their knowledge.

An Aristotelian philosophy of mathematics, on the other hand, sees mathematics as inherently about certain aspects of the real (non-abstract) world — structural and quantitative aspects such as symmetry, continuity and ratio. It sees mathematics as arising from study of those aspects of the world, and pure mathematics as being an intensive study of the harder topics that are found to apply in great generality (or whose motivation has been forgotten over generations).

The talk examines a few of the historical classics in real-world mathematics: Archimedes' derivation of the law of the lever from symmetry, Euler's work on the Bridges of Königsberg, and the exponential model of population growth. In each case it is explained how a naturally-arising problem about a structural aspect of the real world is expressed mathematically so as to reveal the necessities in the world — "Why it must be so". These cases also show that some of the deepest themes of mathematics, such as discrete versus continuous and local versus global, span the pure/applied division.

The talk concludes with a brief look at one contemporary success in incorporating a direct "applied" perspective in teaching, the COMAP Mathematical Contest in Modeling.

Ana Millán Gasca, Francesca Neri Machiaverna* and Emanuela Spagnoletti Zeuli

An experimentation of a learning path on history of mathematics in primary school (Grades 1–5): learning outcomes in mathematics and impact on pupil's human flourishing.

Starting in 2006, topics in the history and anthropology of mathematics have been proposed in mathematics courses for Primary Education at the Roma Tre University MD (Gil Clemente, Millán Gasca 2016). This prompted students to explore the presentation of such historical aspects to children (K–5) during school internships, as well as in-service primary/kindergarten teachers. A tentative learning path and operational sequence was put forward for the introduction of the history of mathematics in Italian primary school (Millán Gasca et al 2017).

A pilot experimentation was carried out from 2018–2023 in a Rome state school class (19 children, including 7 with various special needs, 3 of whom also speak foreign languages), with the first author serving as the maths teacher. A key strategy of this experiment (also for a second experimental path on dance, embodiment and mathematics) was the synergy with children's literature, for which Kieran Egan's story form model was applied for lesson design (Egan 1986). Individual assessment addressed both standard learning outcomes in mathematics and in history (including history of science and technology), as well as "flourishing" personal growth outcomes (Millán Gasca 2016; see Su 2020). Comparing these with the tentative operational sequence, we discuss selection of contents, implementation, and impact regarding the main goals: individual learning, the class life and cohesion, and the rhythm in approaching the mathematical universe from early childhood to the months before entering secondary compulsory school.

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David Guillemette* and Adriano Demattè*

A dialogue on the educator's way to relate to mathematical historical texts.

There is not only one way to read a mathematical historical text. For instance, there is the mathematician who looks at the past from a modern synchronic plan, or the historian who looks at the past from both a synchronic and a diachronic plan. But there is also the educator's (of in-service teachers, of prospective teachers, of secondary school students, of pupils — in short, of "learners") way to read a mathematical historical text which we consider to be different. Indeed, in dialogue with the past, interlocutors assume a certain attitude of *responsibility* that, in educational context, takes a special meaning. Like the historian, educators must accept their vulnerability; that is, to bring out their own inadequacy with respect to the text, posing questions and searching for answers inside the text. More particularly, educators must manage a certain complexity with learners in relation to the past which implies a third role: realizing conditions so that the relation of learners with the author takes place and accompanying them with inclusive gestures in their interpretation enterprise. This role singles out considerably the educator way of relating to the text. On the scene, we have voices from the past, from the classroom, and from a larger sphere of communication — such as those active in the actual sociopolitical environment — that are manifest and that, we think, should be considered. In this communication, with reference to classroom experiments and empirical research results, and focusing on the notion of *responsibility* borrowed from Levinas, we will try to push further the investigation of the educator's way to relate to historical mathematical texts. Our presentation will maintain the form of a dialogue as a report and expansion of our email correspondence.

René Guitart

Learning probabilities by problems and paradoxes: the organization of Joseph Bertrand's textbook (1889).

Joseph Bertrand's textbook *Calcul des probabilités* was published in 1889. This textbook was the result of Bertrand's teaching at the École polytechnique and the Collège de France, and it strongly influenced French mathematicians such as Gaston Darboux, Henri Poincaré and Émile Borel. Thereafter, until the 1940s, it was often referred to.

This textbook is actually a kind of textbook for learning how to learn probability, and today teachers can find in it rich material for building their teaching approach, and in particular to getting rid of possible errors. Moreover, it proposes an approach to learning probabilities by problem solving. Indeed, the author proceeded from numerous "concrete" problems, old and new, in which he illustrated the basic notions of probability calculation, and especially the difficulties and paradoxes to which their misuse can lead.

After a slightly historical reflection on the "laws of chance," in a well-chosen order Betranad addressed the question of equipartition of chances and the composition of probabilities, then the calculation of expectations, the law of large numbers and the ruin of gamblers, the question of the probability of causes, and the law of errors of observation and statistics.

We only illustrate our point with some problems concerning the two most basic questions of equipartition of chances and the ways of combining probabilities from them. In particular, we will examine the famous "Bertrand's paradox."

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Tinne Hoff Kjeldsen* and Uffe Thomas Jankvist

Arguments for history of mathematics in general mathematics education research: A constructive and critical discussion.

The role of history of mathematics in mathematics education and the significance of history for the teaching and learning of mathematics is by now an established research field in didactics of mathematics. Theories and theoretical frameworks for the significance of history of mathematics in mathematics education have been developed. Its practitioners, in particular the HPM group, arrange regularly recurring meetings and conferences, publish a newsletter and have taken the first step towards establishing a peer review journal for publication of research in the field. Theories from other areas of didactics of mathematics are an integral part of much of the research done in the field. In this paper, we look from the other side. We discuss how history of mathematics is used, and with which arguments, when it finds its way into the more general mathematics education research. More specifically, we analyze works from Anna Sfard (1991, 1995), Guy Brousseau (1997) and Guershon Harel and Larry Sowder (2007) with respect to how they use history of mathematics and for what purposes in order to discuss the influence of HPM in more general mathematics education research.

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Jesper Lützen

Hjelmslev's Teaching of his Geometry of Reality.

During the first half of the 20th century the Danish geometer Johannes Hjelmslev (1873–1950) developed what he called a "geometry of reality". It was presented as an alternative to the idealized Euclidean paradigm that had recently been completed by Hilbert. Hjelmslev argued that his geometry of reality was superior to the Euclidean geometry didactically, in practice, and scientifically: didactically, because it was closer to experience and intuition, in practice because it was in accordance with the real geometrical drawing practice of the engineer, and scientifically because it was based on a smaller axiomatic basis than Hilbertian Euclidean geometry but still included the important theorems of ordinary geometry.

It is characteristic of Hjelmslev's geometry of reality that lines have a non-zero width and consequently that two points do not uniquely determine a line: "We have not formulated the requirement that 2 arbitrary points determine a straight line. Indeed, this requirement is in its extreme consequences one of the worst assumptions one has ever introduced in geometry since it is the one that can give rise to the greatest errors."

Hjelmslev based his own teaching of descriptive geometry at the Polytechnic College in Copenhagen on his controversial new ideas and worked hard (but not entirely successfully) to propagate them to the geometry teaching in primary and secondary schools: for 10 years he taught a course on his geometry of reality to the future mathematics teachers at a teacher training college, and he authored a system of textbooks for primary and secondary school based on his realist ideas.

In this research report, I shall explain the main ideas of Hjelmslev's geometry of reality, their origins, and Hjelmslev's attempts to implement them in his own College teaching and in the geometry teaching in primary and secondary schools. I shall also discuss why his ideas had a limited success.

Malgorzata Marciniak

Seeing the development of mathematics education in the light of Kuhn's theory of scientific revolutions.

In 1964 Kuhn formulated a theory of scientific revolutions, using the concept of paradigm shifts, where scientific knowledge undergoes radical changes rather than gradual evolution. His ideas about the incommensurability between competing paradigms and the role of normal science have influenced educational research. Even if this theory was originally developed for natural sciences, its framework can be useful for analyzing the development of other phenomena, in particular mathematics education in various countries. During my talk I will formulate the paradigms of mathematics education and present the historical and modern "revolutions" that motivated their shifts. I will discuss suitable examples of the pivoting moments from the past that shaped our Western modern education, for example the introduction of compulsory education, the introduction of secular education, and making public education available to everybody. More modern education reforms that impacted teaching and learning include "New Math", or the "No Child Left Behind Act" which was replaced later by the "Every Student Succeeds Act". I will explain their motivations, pros and cons in the light of the shifts of the previously defined paradigms.

In the context of Indonesian education, the paradigm shifts went through different routes. Being currently the fourth largest (after China, India, and USA) education system with over 50 million students and 3 million teachers, Indonesian education was nationalized in 1945 and later went through transformations in the sixties and the seventies due to political changes. Regardless of multiple reforms, the schools remain not only under the responsibility of the *Ministry of Education, Culture, Research, and Technology,* but as well under the *Ministry of Religious Affairs*. Facing a variety of geographical, organizational, and ethnic challenges, Indonesian mathematics education emphasizing ethnomathematics and realistic mathematics education was influenced by the ideas of Hans Freudenthal.

A part of the presentation will be devoted to the very recent sudden transformations worldwide that happened during the pandemic. In the context of previously defined paradigms, I will discuss the shifts and challenges faced by teachers from Indonesia in the context of journal submissions sent by researchers from that country.

Kay Owens,* Vagi Bino* and Charly Muke*

The Development of Neocolonialism in Papua New Guinea.

Papua New Guinea was first colonized by Germany in the north and Britain, followed by Australia, in the south. After World War 1, both Territories, including education, were managed by Australia. Mathematics exams developed during the 1900s prior to independence, and exams have continued making mathematics learning in school highly valued as exams are gatekeepers for continuing education. However, at Independence the culturally responsive education plan proposed by Papua New Guineans was rejected for an expatriate devised plan although it was partially recognized that education had to prepare students for life in the village and not just for higher education. At Independence, from a late starting block, universities produced graduates and colleges produced teachers. Gradually national curricula for education and teacher education developed encouraging standards of education. It took 10 years for another plan focusing on maintaining culture, universal education, and gender equity to be put forward. Meanwhile languages were being lost and there were inadequate funds in place to make the new plan viable for village schools with mathematics in vernacular languages reflecting culture. It also came with school restructuring that was quite disruptive, and an education that was still terminated by exams or fees. This reform has been reversed by a recent government. The question remains whether it is possible to reverse the downward trend away from culturally recognized mathematics and responsive mathematics education given the current status of organization of education, general country milieu, and increasing loss of language. Glimpses of successful programs and teacher quality suggest that families will need to recognize their cultural mathematics and encourage it for their own family and place.

Hélder Pinto* and Helmut Malonek

The proofs of Euclid on GeoGebra, a step-by-step visualization.

In this presentation, we will present an example of how to use technology, specifically GeoGebra®, to approach a significant book in the History of Mathematics: Euclid's *Elements* (particularly, Book I which ends with the well-known Pythagorean Theorem and its converse). Using this dynamic geometry software allows new approaches to Euclid's proofs, enabling students to have access to step-by-step constructed images, rather than just a static final image as seen in a book or PDF. These GeoGebra® applets use a color scheme to facilitate the understanding of what is being used at each moment of the proof: in black are the initial data, in red is what is being displayed, in green is what is being used to deduce what is in red, and in blue are previous steps of the proof. Additionally, we will show how this approach can be useful in teaching, showing students the "power" of mathematical proofs and their justifications. The main idea of this approach is to give students a (visual and interactive) tool for better understanding how the arguments used in each step of the Euclid's proofs work and to understand the need to justify each statement throughout the proof. It also aims to show the cumulative nature of mathematical knowledge, since most proofs used previous propositions (or facts that were initially assumed to be true, i.e., Postulates and Common Notions). It is also intended that students become familiar with the concept of mathematical proof and to know better some of the geometry that is still used today in school mathematics.

This is an ongoing work that aims, in the end, to have all the demonstrations from Book I accessible on GeoGebra® for free use. As an illustrative example for this work, the «step 6» of the proof of Elements I,1 (*To construct an equilateral triangle on a given finite straight line*) is reproduced below (Fig. 1).

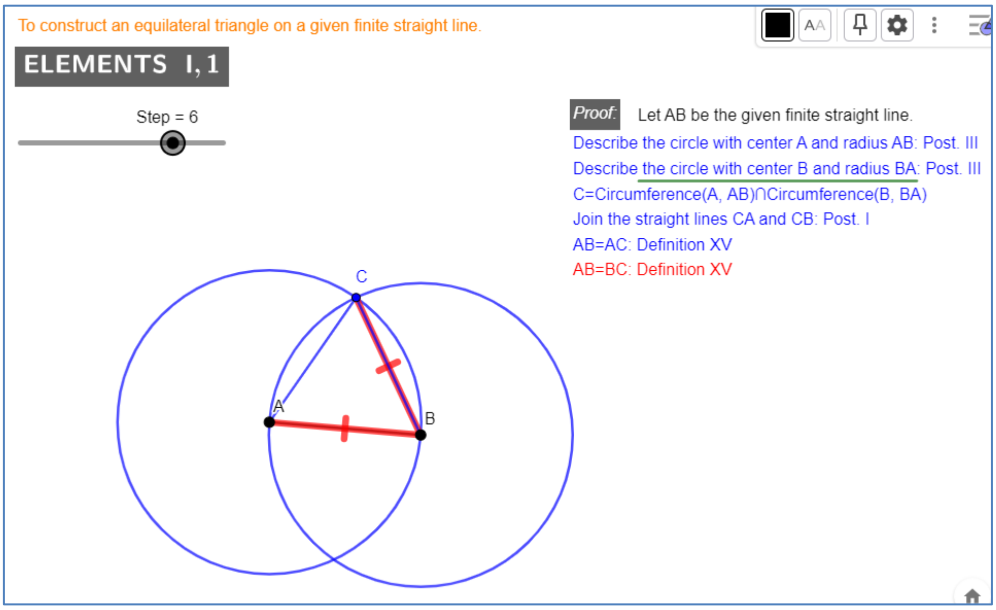

Fig. 1: «Step 6» from Euclid's proof of Elements I, 1

François Plantade

Jules Houël (1823–1886): from teaching geometry in high-schools to resolving the question of the independence of Euclid's postulate in France.

Jules Houël (1823–1886) was a French mathematician and astronomer, from an old Normand Protestant family. After studying at the École Normale, he taught Euclidean geometry at various high schools from 1846 to 1856 (in Bourges, Bordeaux, Pau, Alençon, Caen). On that topic, Houël was a fervent defender of a geometrically pure teaching method and for this reason he rejected the "hybrid" method developed in the Legendre's *Éléments de géométrie*. After this period, he taught infinitesimal calculus at the Bordeaux Faculty of Science from 1859 to 1884. Nevertheless, the question of the foundations of classical geometry was a constant subject of reflection to him: as early as 1863, he published several papers/memoirs on the reformulation of Euclid's postulates, such as *Essai d'une exposition rationnelle des principes fondamentaux de la géométrie élémentaire* and *Essai critique sur les principes fondamentaux de la géométrie élémentaire, ou Commentaire sur les XXXII premières propositions d'Euclide*. Houël's deep motivations for the foundations of geometry were linked to his experience in high school teaching. He had a point of view that was similar to those of Newton (for the experimental side of axioms and movement), and of Euler and D'Alembert (for the importance of the assumption of "impenetrability"). He was also interested in the question of the independence of the parallels postulate, which led him to rediscover, translate and publish the work of Lobachevski, Bolyai, Riemann, Beltrami, etc. on non-Euclidean geometries, starting in 1866 in the *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*. The SSPN, a small provincial society until 1866, extended its branches outside France very quickly, as a result of Houël's work as an archivist and contributor. In fact, in 1866, the SSPN was in an exchange relationship with 20 learned societies, in 1869 with a little over 80, and in 1875 with a little over 170 societies, many of them abroad. Houël also brought the theory of the foundations of geometry onto an epistemological ground; he studied the synthetic nature of geometry while opposing the purely axiomatic vision. Moreover, the conception based on the axioms of elementary geometry and his ideas on the independence of the postulate of parallels did not vary between 1863 and 1883. The Carton affair at the Paris Academy of Sciences in 1869 precipitated the questioning of the postulate of parallels, and, finally, Houël concluded the question in the *Note sur l'impossibilité de démontrer par une construction plane le principe des parallèles*. Our presentation will be based on geometric teaching in high schools in France about 1850, and will outline the epistemological and historical milestones in Houël's reflections on the foundations of classical geometry. Houël's connections with the foundations of geometry and non-Euclidean geometries will also be discussed, in particular with Italian mathematicians (Battaglini, Beltrami, Cremona, etc.). *Nota bene*. Houël's view of the foundations of geometry has nothing to do with Hilbert's *Grundlagen*.

Claire Poh Hwee Sim

Threads of Knowledge: Crafting a Cultural Tapestry in Mathematics Education.

Consider the mathematics education landscape as an interwoven fabric, where threads of cultural tradition and historical legacies are skillfully entwined to create a tapestry of learning. Each thread contributes a unique texture, reflecting the cultural traditions and historical legacies that shape the mathematics education narrative. The patterns formed by these threads serve as a testament to the interplay between indigenous knowledge and pedagogical creativity that shapes the fabric of an inclusive mathematics curriculum.

Embracing diversity and inclusivity in mathematics education underscores the need for nuanced, contextually relevant, and culturally sensitive approaches to teaching mathematics. Drawing upon the words of artist David Mowaljarlai, 'pattern thinking is Aboriginal thinking,' each individual is inextricably connected to a knowledge system. Mathematics itself is not a monolithic construct; it is created by many cultures around the world, each imbuing it with their unique perspectives, symbols, and meanings. It is about shared concepts and communal belonging that resonate with the Aboriginal philosophy and knowledge system, that everyone within the system belongs and holds agency and responsibility to uphold and fortify it. Regrettably, Western/Eurocentric teaching models in mathematics are often fixated on abstract concepts, making it difficult for students to connect these ideas to their lived experiences. This disconnect can impede their engagement with and understanding of mathematics.

The Goompi model, devised by Dr. Chris Matthews, offers a departure from conventional approaches to teaching mathematics, emphasising creativity and students' personalised expressions of mathematical concepts. At the heart of this model lies the learner's perspective, allowing students to engage with mathematics in ways that resonate with their own experiences and cultural backgrounds. The Goompi model addresses this by creating new pedagogies that foreground culture and cultural expression, such as storytelling and dance. By incorporating these innovative approaches, students can develop a deeper understanding of mathematics while also connecting it to their cultural heritage.

Research findings further underscore the transformative impact of the Goompi model on students' learning experiences. By fostering a positive sense of Aboriginal identity and offering a framework for culturally rich resources and learning environments, students are provided with opportunities to engage with mathematics in ways that are both meaningful and relevant to their lives.

Challenges often spark new ideas, and obstacles provide valuable opportunities to reassess our understanding of mathematics education. At its core, mathematics is deeply entrenched in our culture, akin to the threads intricately woven into the fabric of our society.

Luis Puig

Errors dealing with the negative in solving quadratic equations. An episode in the history of algebra and its teaching.

The codex Dresden C80 belonged to Johannes Widmann who used it when he lectured on algebra at the University of Leipzig in 1486. Among the documents it contains there is a *Latin Algebra* (Wappler, 1887), in which the rule for solving one of the types of second degree equations ($ax^2 + c = bx$, in current terminology) includes an error. The rule is equivalent to

$$\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

what is correct. The error appears in a prompt at the end of the rule: "if you can't subtract [c/a], you are allowed to add" (Wappler, 1887, p. 14).

This authorization to add instead of subtract also appears in Bombelli's *L'Algebra*, 1572. After having solved the equation $x^2 + 20 = 8x$ by giving its two imaginary solutions, Bombelli says that "there is another sophistic method, that since 20 cannot be subtracted from 16, it is added", and he obtains a third solution of the equation, 10, and adds "and this 10 is minus" (Bombelli, 1572, pp. 262-263), without any explanation.

This is not the only error in the rules for solving this type of equation. In a manuscript before 1504, *Die "Algebra" des Initius Algebras* (Curtze, 1902), the rule is stated correctly, but when applied to an example, the order of subtraction is wrongly reversed.

$$\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \pm \frac{b}{2a}$$

And in Adam Ries' manuscript *Die Coss* (1527), edited by Rüdiger, Rainer & Folkerts (2023), the rule itself is stated with the subtraction erroneously reversed.

Both errors appear in Marco Aurel's *Arithmetica Algebratica*, published in Spanish in 1552. He uses the erroneous rules in an example and a problem and explains how to obtain the correct result from the erroneous one or how to interpret the wrong result as correct. Its influence on other books written in Spanish, and especially the success of Pérez de Moya's *Arithmetica Practica y Especulativa*, published in 1562 and reprinted a lot of times until 1798, made these errors last long in Spain.

In this communication we study the statements of these erroneous rules, their eventual use in examples and problems, the justifications for the way of obtaining the correct result from the incorrect one or interpreting it as correct, and their dissemination.

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Agathe Rolland and Renaud Chorlay*

Expectations regarding French prospective teacher's knowledge in group theory: A historical survey.

History of mathematics and history of mathematics education have long been identified as valuable sources for didactical reflection ((Artigue (1991), (Chorlay & Hosson, 2016), (Chorlay, Clark & Tzanakis, 2022)). The aim of this paper is to present elements of history of education that provide background information for a PhD dissertation (A. Rolland) on the links between mathematical knowledge acquired at university and its use by secondary school teachers — a topic known as "Klein's second discontinuity". The specific topic chosen is that of group theory. The final aim of the PhD is to produce a capstone course, i.e. a didactically engineered course for pre-service teachers that would be backed up by general methodological and theoretical reflections on Klein's second discontinuity.

One of the challenges is to understand — and to make prospective teachers understand — the role of abstract theories whose link with the contents taught in secondary education is not transparent. Rather than attempting to determine *a priori* (through mathematical and/or didactic reflection) what this role is in the case of groups, we will endeavour, through a historical study of one of the main French competitive examinations for teacher recruitment (*agrégation*), to identify the expectations and motivations of the hiring authorities. The choice of the period of study — 1950–1990 — allows us to investigate variations in these expectations, showing that they may have been significant even before and after the modern maths period (de Bock, 2023) (Gispert, 2023).

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Jorge Soto-Andrade, * Dandan Sun and Daniela Díaz-Rojas

Avatars of (random) numbers in the history and experimental epistemology of mathematics.

We explore several intertwined threads related to various avatars (or metaphors) for the concept of number, from the viewpoint of cognitive archaeology (in the sense of Overmann et al.) and experimental epistemology of mathematics (in the sense of Brousseau), later dubbed didactics of mathematics). First, we recall Shao Yong's synthetic and dynamic visions of Yi Jing (cf. his Xian Tian diagrams), where random hexagrams appear as the outcome of ascending random walks in a six-generation rooted binary tree. Interpreting hexagrams as the binary expansions of numbers from 0 to 63, à la Leibnitz, we come to metaphorise a random number from 0 to 63 as a random walk. Remarkably, a natural extension of this process leads to Hensel's 2-adic integers, nine centuries later, as well as to the contemporary stochastic discrete models of the multiverse, in "eternal inflation" in cosmology. Second, we show how random numbers, now embodied as *dice* instead of random walks, provide a natural and concrete model for the non-transitive orderings in the classical Chinese Five Element Cycles (involving water, fire, metal, wood, earth). Third, in the spirit of cognitive archaeology, we trace back the history of dice in humankind, in parallel with the instantiations, icons and symbols for numbers. From the viewpoint of experimental epistemology of mathematics, we discuss how dice, taken at face value, may afford a place value number system *without zero* (or even a blank space), which affords an unambiguous recording of quantities (e. g. 12 = 1 and 6) and addition with carry over, in a friendlier way than our decimal number system does.

We ponder upon the fact that although zero is not *necessary* to have a positional system, such a dice based positional number system does not seem to have arisen anywhere on earth, although we observe its emergence in our classrooms, under minimal prompting. We also discuss the emergence of Shao Yong's synthetic insights under analogous *a-didactic* conditions and the role of epoché and phenomenological reduction in experimental epistemologic activities where we try to elicit the arising of certain concepts, procedures or metaphors, in contemporary circumstances. Finally, we discuss the intertwining of cognitive archaeology and experimental epistemology in the case of mathematics and its teaching and learning, with an eye on the Whiggish pitfalls lurking in our approach. Particularly, we argue in favour of a random walk model for the rhizomatic development of mathematics in humankind.

Noriko Tanaka

How to learn Japanese mathematics "WASAN" in the Edo period (1603–1868).

There were several schools of mathematics in Japan during the Edo period (1603–1868). Each school had its way of writing formulas and expressing mathematics differently. The most famous and popular school was the Seki school of the famous mathematician Seki Takakazu.

There are few written records of how students learned mathematics during the Edo period, but one small school, the Shisei-Sanka school, has left behind a rare book describing how students learned. It says, for example,

> When you hold up a question on a tablet wooden board to a particular student, ask that student by name. When you ask the whole group a question, you should write: 'Ask this to everyone.' Students should have approximately one to two months to solve and answer the question, depending on the complexity of the answer the question requires.

Although most of the problems were about plane geometry, they were complicated and time-consuming to solve. It was also stated that

> Students are strictly forbidden to discuss the problems from our school with those of other schools. Therefore, we do not allow the questions of other schools to be included in the wooden boards of this school.

This instruction shows that mathematics was kept secret by each school.

When students solved an excellent mathematical problem, they thanked the gods or Buddha, wrote the problem and the formula for solving it on a wooden panel called a SANGAKU (arithmetic tablet), and dedicated it to a shrine or temple. The geometrical figures on the SANGAKU were painted in colorful colors. The SANGAKU are still preserved in shrines and temples in Japan today.

I want to share the Japanese way of learning mathematics in the Edo period, which is not well-known internationally. I will introduce the mathematics studied there and the problems published in the SANGAKU to show how the Japanese people were familiar with mathematics then. The many versions of the Edo period mathematics book *Jinko-ki* became a huge bestseller. Western mathematics began to spread in Japan at the end of the Edo period, and it was taught in the new schools at the Meiji Era (1868–1912). The grounding in studying Japanese mathematics is thought to have been a factor in the rapid penetration of Western mathematics.

Jia-Ming Ying, * Chia-Jui Hsieh and Lin-Chieh Tsai

Influences of a Liberal-Art Course about East-Asian Mathematical Culture on University Students' Mathematics Beliefs.

This paper examines the impacts of a liberal-art course about East-Asian mathematical cultures on university students' mathematics beliefs. The course, taught in a comprehensive university in Taiwan, explored pre-modern East-Asian mathematics, focusing on traditional Korean culture and mathematics, which have deep links to its Chinese counterparts. This course covered key theories, methods, and problems of pre-modern East-Asian mathematics, highlighting unique reasoning patterns and possible applications in fields such as astronomy and civil engineering. In addition to Chinese treatises, this course exposed students to some Korean scientific artefacts, such as the *Cheomseongdae* ('star-gazing tower'), counting rods, and mathematical texts containing problems related to government functions in pre-modern East Asia. We used a survey research method, with quantitative and qualitative tools. The quantitative tool was a twenty-seven-item Likert-scale questionnaire on mathematics beliefs, which was designed to examine students' beliefs in the dimensions of the 'nature' and 'values' of mathematics; the qualitative tool was students' after-class reflexions. The two tools were analysed with descriptive and inferential statistics, and content analysis, respectively. A total of forty-six students from six different colleges (Science, Engineering, Nuclear Science, Humanities and Social Sciences, Electrical Engineering and Computer Sciences, and Education) of that university took the course, wrote several reflexions, and finished both the pre-test and post-test of the questionnaire. Quantitative results show that, generally speaking, for the dimension of the nature of mathematics, students had a more diversified understanding of the justification of mathematical knowledge; for the dimension of the values of mathematics, students tended to agree more on the links between mathematics and traditional cultures, humanities, and real-world applications. Individual students' changes between the pre-test and post-test have also been compared, and it could be seen that most students were inclined to more diversified and culturally-oriented beliefs about mathematics. Qualitative data were used to triangulate our findings from the quantitative data. In general, research results show that the liberal-arts course about East-Asian mathematical cultures has a similar impact with earlier studies on university students' mathematics beliefs in terms of the nature of mathematics, in which students, though highly approve of the role of logic in mathematics, also confuse the context of discovery with that of justification in mathematics; results also show that after taking this kind of course students' beliefs in terms of the values of mathematics have a stronger tendency that mathematics can be linked to real-world applications, traditional cultures, and humanities and social sciences.

Yixuan Zhu

The Concepts of Curves and Equations in Early American & British Textbooks on Analytic Geometry.

Curves and equations, also known as the Cartesian onnection, is an important and fundamental concept in plane analytic geometry, but it has been shown that students lack a rigorous understanding of it. Focusing on the concept of curves and equations, we investigated 84 kinds of American and British analytic geometry textbooks published from 1826 to 1963. The study found that three types of non-rigorous definitions were prevalent in early textbooks published before the 20th century. The first type merely recognized a connection between curves and equations, but did not express the nature of the strict correspondence. The second type, although intentionally emphasizing that each point needs to be satisfied, focused only on one of the directions, rather than the mutual correspondence. The third category of non-rigorous definitions confused the concepts of curves with functions. Definitions of curves and equations have become increasingly rigorous in post-20th century textbooks, and these rigorous definitions can be divided into descriptive definitions, definitions based on the concept of set, and definitions based on sufficient and necessary conditions. It is noteworthy that the second category of non-rigorous definitions was not eliminated until the middle of the 20th century. Epistemological barriers in the historical development of the curves and equations concepts can become cognitive barriers for students in the classroom as well. Teaching strategies can be developed to guide students in recognizing non-rigorous definitions, leading them through a historical reconstruction of how concepts related to curves and equations have evolved. This approach facilitates a natural progression from abstract to rigorous understanding. In addition, early textbooks not only provide a variety of methods for verifying the Cartesian connection, but also distinguish the similarities and differences between the concepts of curves and equations in Cartesian and polar coordinate systems, all of which provide abundant material for teaching.

1-HOUR WORKSHOPS

Jean Michel Delire

How to construct and use instruments with the pupils, so that they appreciate what mathematics are for: description of two projects (2021–22 and 2023–24) in a Brussels secondary 'active school'.

During two academic years, we proposed to the Lycée intégral Roger Lallemand (LiRL), a Brussels secondary school based on active pedagogy, two different projects relying mathematics with surveying and music.

The LiRL works on three weeks ('triplette') multidisciplinary themes, like *Climate*, *Navigation*, *Coal*, … or *Angela Davis*, for instance. During the triplette mornings ('modules'), the pupils attend courses in different fields, and in the afternoons ('workshops'), they have the opportunity, thanks to the existence of workshop-classrooms in the LiRL, to build their own instruments, with the help of a technology teacher.

Our first project, titled *En vrai, la géométrie ça sert à quoi ?* and subsidized by the Brussels Regional Government (Cocof) after a contest called *La culture a de la classe*, aimed at teaching third secondary grade (14 years old) pupils how some ancient instruments were made and used in order to elaborate precise maps and plans. The utility of these instruments (Geometric Square, Sector) relies on proportions, which were studied during the mathematical modules. Other modules (French, History) investigated the origins of these instruments, of which some were developped in Belgium, by Michel Coignet (1549–1623) in Antwerp and by Gemma Frisius (1508–1555) in Louvain, for instance. The argument of the triplette was more generally defined as *Reconstruction after a disaster*, and the final task of the pupils was to measure with the instruments a square (Place Morichar) nearby their school, and to imagine how to rebuild it.

The second project, also subsidized by the Cocof, was titled *Accords et ondes*, and aimed at teaching fifth secondary grade (16 years old) pupils wave characteristics and their relation to the musical theory. Here, the morning modules were devoted to the study of physical and mathematical concepts, such as wave lengths, frequences, speeds in different environments, or proportions, geometrical progressions, roots and their properties. In the afternoon workshops, the pupils tested wave propagation, calculated the speed of sound, listened to sounds of different frequences, in tune or not, with the help of a musician participating in the project. They also construct a simple instrument, called *Epinette des Vosges*, comparable to the *Dulcimer*, in order to understand the importance of the musical scales and intervals.

During the workshop, we will distribute to the participants cardboard models of the instruments and show some pictures and films shot during the projects. We will also present historical texts (translated from French) that induce the pupils, and the participants at the workshop, to use the instruments and appreciate mathematics.

Hanna Nadim Haydar* and Burcu Durmaz*

Teaching Mathematizing Through Stories from the History of Mathematics: Promoting Culturally Responsive and Sustainable Mathematics Education.

The session will start with a presentation showcasing the use of a children's story as a context for mathematizing (Freudenthal, 1991). We will model the teacher's role in initiating a lesson through storytelling. Participants will then actively engage by responding to questions related to the story and solving embedded mathematical tasks. This will serve as a foundation for discussing key aspects of our methodology (Haydar & Durmaz, 2022) focusing on the integration of mathematics and storytelling (Sims Bishop, 2011), the utilization of contexts for mathematizing, and the incorporation of the history of mathematics in a framework of culturally responsive/sustainable pedagogy (Djebbar et al., 2009).

In the second half, we will work in two groups and analyze two instructional units:

Al Kindi and Cryptography:
The unit begins with notice and wonder routines introducing code-breaking, followed by an online game. A historical storytelling context delves into Julius Caesar's coded messages during wars, leading to mini-lessons on inverse operations and functional rules. The literacy connection includes a reading on the House of Wisdom, featuring the Arab philosopher and mathematician Al Kindi and his work on cryptography. This sets the stage for an investigation guiding students to reconstruct Al Kindi's frequency analysis method. Social studies connections explore the history of Baghdad and the scientific contributions from the House of Wisdom.

Ibn Al Haytham and Visual Proofs:
The unit begins with a children's story about Cubey Cake, using a birthday cake and dice context to introduce a guided proof of adding consecutive whole numbers 1 to 6. A task follows on constructing Ibn Al Haytham's visual proof using Cuisenaire rods investigations. A video story narrating Ibn Al Haytham's biography leads to interdisciplinary connections around his journey and imprisonment in Egypt and his seminal work on Optics. The subsequent mathematical investigation leads to generalizing the formula for the sum of consecutive numbers and culminates with the story of Gauss.

We will conclude with participants' feedback and general discussion.

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Rainer Kaenders

How can the history of the existence of fourth proportionals from Eudoxos via Omar Khayyam and Nasir al-Din al-Tusi to Isaac Newton foster a modern mathematical number concept?

A concept development towards a modern understanding of real numbers is explored using the historical genetic method. First, we consider the concept of the fourth proportional (given magnitudes *a* and *b* of the same kind and another magnitude *d*, then there is a magnitude *c* of the same kind as *d*, such that *a:b=c:d*) at the hand of excerpts from book V, attributed to Eudoxos, in Euclid's *Elements*. Then we discover and discuss situations in contemporary school textbooks in which the existence of fourth proportionals is assumed and used. We look at these textbooks through the eyes of Eudoxos and Omar Khayyam (1048–1131). In Euclid's book V, we study definition 5 of the equality of two ratios and compare it with the "antiphairetic" definition based on the Euclidean algorithm. With Khayyam, for us (positive real) numbers are nothing but proportions of magnitudes that, surprisingly, can be calculated with if at least the calculation laws are proven, which Euclid and Khayyam partly did. The proof of this is in Euclid's book V and later in Nasir al-Din al-Tusi (1201–1274). In the early modern period, the concept of number as a proportion of magnitudes was newly clarified by Isaac Newton (1642–1726) and later attributed to him (cf. Frege, p. 25). The existence of the fourth proportional can serve as "Grundvorstellung" (mental model) for the arithmetic of real numbers, which we try to develop together with the participants along the historical perspectives such that the educational potential becomes visible.

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Hélder Pinto

The importance of History of Mathematics in the classroom (by ChatGPT): a first reflection about the use of ChatGPT in HPM.

On the 15th of January (2024), a CNN headline pointed out that « 'Jobs may disappear': Nearly 40% of global employment could be disrupted by AI » (statement from the IMF [1]). In Mathematics education, the advancement and widespread application of AI are expected to introduce new challenges, which are currently under investigation and will require some time to be thoroughly comprehended [2, 3]. The question is to know if historians of mathematics will be among these newly unemployed? And the teachers of mathematics? And the teachers that use History of Mathematics? Will they be replaced by AI too? It should be noted that the debate on the implications that AI has/will have on mathematics education is still in its infancy, although it is already recognized by some authors that "ChatGPT is a useful educational tool, but caution is needed when using it, and guidelines for safe usage should be developed" [4].

As an experiment, we ask ChatGPT to do an abstract for HPM 2024 with only six iterations:

- « Made me an abstract about the importance of History of Mathematics in classroom (250-400 words) »;
- « Can you add some references? »;
- « I want the references inside the text and a list in the final. »;
- « Are there critics about the use of the history of mathematics in the classroom? »;
- « Can you insert these critics in the abstract above? »; and
- « With more references in the text please ».

The result is clearly not perfect (for instance, one given reference does not exist), but it is a good example of what a student could do in just a few minutes. Below is the beginning of ChatGPT's abstract:

> *The study of the history of mathematics holds immense importance in the classroom, serving as a powerful tool to enhance students' understanding of mathematical concepts and fostering a deeper appreciation for the discipline (Dunham, 1990; Stedall, 2012). However, this approach is not without its critics, who highlight potential challenges associated with its implementation.*
>
> *While the integration of historical narratives into the curriculum is intended to enrich the learning experience and nurture critical thinking skills (Dunham, 1990), some critics argue that the historical content might not always align with modern curriculum goals and the practical needs of students. They contend that time spent on historical context could be better utilized for covering more contemporary and applicable mathematical concepts.*

In this workshop we intend that the participants use ChatGPT to create materials for teaching and for the History of Mathematics, in order to provoke reflection on the potentialities and dangers that may arise for the HPM field from this kind of applications. It is intended to think about the major questions about the use of AI in math education, particularly, about the conceptualization of historical knowledge as well as the role of the teacher in historically oriented mathematics education.

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2-HOUR WORKSHOPS

Silvia Benvenuti

The mental telescope: the non-Euclidean geometry case study.

The story of the birth of non-Euclidean geometries is a very interesting chapter in the history of mathematics, for a number of reasons. First of all, the current systematization was reached only by a troubled path that lasted many centuries, which suggests that perhaps there was an epistemological obstacle hiding, which the historical perspective highlights. Moreover, the history of the development of these new geometries is fascinating because it provides an example of how, although starting from very remote roots, mathematics is a subject in constant evolution, alive and susceptible to change, against the stereotype that sees it instead as a "dead" topic. Finally, analysing the development of this branch of geometry serves to combat the stereotype that mathematics is the creation of isolated geniuses, presenting a clear example of how it is instead a wonderful collective cultural creation.

Our intent is to show that non-Euclidean geometries can be a tool to promote the understanding of the modern axiomatic method in mathematics, to stimulate students' aptitudes to logical thinking and to allow students to consolidate the knowledge of Euclidean geometry by developing it in a critical way.

Planned structure of the Workshop:

- Introduction to spherical geometry (laboratorial session).
- Formalization: what is non-Euclidean geometry (with historical glance), axiomatization of geometries.
- The mental telescope: how non-Euclidean mathematical ideas have provided tools for physics and fertilised art, showing how the interdisciplinary approach is successful not only in teaching, but also in the development of culture as a whole.

Polystyrene balls, rubber bands, drawing pins, small wooden trains, felt-tip pens, inflatable balls (all provided by the speaker).

Piotr Błaszczyk* and Anna Petiurenko*

Newton's De Analysi vs Fundamental Theorem of Calculus.

In *De Analysi* [1], [2], Newton derives three primary achievements of modern calculus: the area under the curve $y(x) = x^{\frac{m}{n}}$ equals $\frac{n}{m+n}x^{\frac{m+n}{n}}$ (Rule I), the power series of arcsine, and the power series of sine.

Two further rules introduced without proof reinforce Rule I. Rules II and III state that the area under finitely or infinitely many curves equals the sum of areas under each curve.

The standard interpretation of *De Analysi* runs through calculus: adopting the Riemann integral, it presents Rule I as the Fundamental Theorem of Calculus $\left(\int_0^x f(g)dt\right)' = f(x)$. Accordingly, term-by-term integration of series explains Rule III. However, this interpretation does not correspond to the argument's structure regarding the series of arcsine and sine. In calculus, one first expands the series of sine and then gets the expansion of arcsine by the theorem on the inverse function derivative. On the contrary, Newton finds the power series of arcsine first and then the series of sine. The core of this difference is that Newton does not apply the derivative or limit concept.

We present actual techniques applied by Newton, namely Euclidean proportion, indivisibles, 'infinitely close' relation, and formal power series.

Regarding the study of arcsine and sine, we present Newton's results in a broader context, ranging from Ptolemy to Euler.

During the workshop, we will walk participants through Newton's infinitesimals and formal power series techniques.

The workshop rests on the first English translation of *De Analysi* [1]. We will provide its electronic version in advance. One can also browse it through the Internet Archive digital library.

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Silu Liu* and Renaud Chorlay*

Selecting episodes shedding light on the history of the function concept: historical and didactical analyses of a lesson-study in grade 10.

The design of didactic situations aiming to take students along a cognitive path of amendment and expansion of their concept of "function" has already been studied in an HPM context (Kjeldsen & Petersen, 2014). In 2018, a different session — with roughly the same goals — was designed and implemented in the context of the Shanghai HPM-studio, in the lesson-study format (Liu, Shen & Wang, 2019; Wang & Shen, 2020). The goal of the workshop is to further describe, analyze, and discuss the various choices made by the participants in this lesson-study.

More specifically, we plan to:

- Present a selection of historical episodes — some well-known, some less well-known — which could be used in the design of a session aiming to take students along the expected cognitive path.
- Offer a didactical analysis of the expected benefits of an integration of accounts of these episodes in the session design.
- Share some empirical data collected during the implementation of the session in 2018 so as to discuss the choices made by the stakeholders in the lesson-study, as well as their actual effects on students.

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Pietro Milici,* Cinzia Cerroni, Benedetto Di Paola and Anna Ruggeri

Touch, experience, and re-think calculus with history-based manipulatives.

In continuity with Descartes, the role of geometrical machines is at the basis of Leibniz's conception of infinitesimal analysis. The main characteristic of these machines is to solve inverse tangent problems, i.e. to construct curves given the properties of their tangents. The proposed workshop aims to relive some aspects of this at the intersection of the history of calculus and the history of scientific instruments. Specifically, we propose to introduce a geometric-mechanical artefact realized with 3D printing and laser cutting process. This history-based manipulative presents some new peculiarities (the first author patented it): it is designed with particular attention to simplicity and to hide nothing from users, and its components can be assembled in various ways (to get an idea, see www.machines4math.com). Its adoption allows the experience of significant historical steps not only through words but through hands-on activities: from Leibniz's early conception of calculus (tractional motion, 17th century) to the Enlightenment (demonstration machines, 18th century) and the realisation of commercial instruments (integraphs, 19th–20th century). During the workshop, we propose activities on the mechanical implementation of the tangent to a curve (direct and inverse problem), on the fundamental theorem of calculus (derivatives and antiderivatives), but also activities aimed at the construction of transcendental curves by the solution of differential equations (dynamic slope-field solvers). The attendees will have the chance to touch/experience calculus and to re-think it, thanks also to the epistemological contents involved in the recalled historical approach. In the final part of the workshop, we will discuss some results obtained with students engaged in the construction of calculus knowledge mediated by our artefact.

We will provide several copies of the proposed geometric-mechanical artefact to be used by attendees. Suitable printed worksheets will be integrated with slides on the related historical/epistemological contents and with digital simulations for the various activities.

<u>POSTERS</u>

Carmen León-Mantero, * José Carlos Casas-Rosal* and María José Madrid

The infinitesimal calculus in textbooks published in Spain before its incorporation into the curriculum of secondary education.

The research in the field of learning and teaching calculus focuses on trends such as cognitive development or task design with particular attention to the concepts of limits, derivatives, and integrals (Bressoud et al., 2016). Studies on calculus concepts approached through the History of Mathematics Education can help to face this problem since conceptual difficulties with teaching mathematics often correspond with historical periods of conceptual crisis in mathematics (Heeffer, 2006).

This research project aims to analyze the evolution of the contents of calculus and its teaching through teaching textbooks. The period selected is between the introduction of this branch of mathematics in Spain as a scientific discipline, that is, at the beginning of the 18th century, until the implementation of the first study plan that included it among its teachings, the Plan Pidal of 1845.

It is an exploratory, descriptive, and qualitative historical investigation focused on analyzing old textbooks from the perspective of the History of Mathematics Education. It is approached through the historical research method proposed by Ruíz (1976) and the content analysis technique proposed by Maz (2005).

The results of this project may be of interest to improving the training of mathematics teachers. They allow us to identify how, when, and why the difficulties students typically encounter when dealing with calculus-related topics in the classroom arose. It will also allow us to learn about the scientific approaches and teaching strategies used in the past to address these difficulties.

The selected study period coincides with the influence of recognized religious and military and, later, civil institutions dedicated to training engineers and professionals in developing and disseminating calculus from the mid-18th century to the beginning of the 19th century. This period also saw the consolidation of calculus in the rest of Europe. Finally, the calculus textbooks published in Spain during this period contain misconceptions, such as the non-consideration of negative algebraic solutions. Therefore, it is relevant to know how and when the advances achieved in Europe were introduced in the teaching of calculus in Spain and whether the same misconceptions continue to be made over the selected period.

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María José Madrid, Carmen León-Mantero* and José Carlos Casas-Rosal*

Exploring the teaching of geometry for children in 19th century Spanish educational press: the case of 12-year-old Carlitos teaching to his friends.

Educational journals targeting children emerged in Spain during the 19th century and their study allow us to extend the existing knowledge base about mathematics education in Spain at the time. One of these journals was *Los Niños: revista de educación y recreo* (*Children: journal of education and leisure*), published in Madrid, directed by Carlos Frontaura and featuring contributions from various writers. Among these contributions are 32 articles published between 1871 and 1872 by Eduardo Tuiller which focus on geometry for children. The aim of our study is to know more about the teaching of geometry at the time through the analysis of these publications.

To achieve this, we conducted a descriptive and exploratory research of a historical-mathematical nature, employing content analysis techniques for analyzing texts from the past. The journal's issues were accessed through the Virtual Library of Historical Press, provided by the Spanish Ministry of Culture and Sports.

The results show how the author presents geometry through a narrative featuring a 12-year-old boy named Carlitos, who teaches geometry to his friends in the garden of the house of one of them. The author justifies this by explaining that Carlitos' friends have exams and are falling behind in geometry, whereas Carlitos is ahead. Consequently, they agree that instead of engaging in other games, Carlitos will teach lessons on this subject in the afternoon.

In each lesson, Carlitos explains various geometric concepts, both in two-dimensional and three-dimensional space, including: length, lines, straight lines, angles, triangles, polygons, quadrilaterals, parallelograms, circumferences, circles, areas of polygons, planes, polyhedra, prisms, pyramids, cones, cylinders, spheres, or volumes.

Throughout the articles, the different concepts are presented as dialogues among Carlitos and his friends. To explain the concepts, Carlitos uses various materials such as pieces of rope, paper, or wood, rods, sticks, rings, etc. Additionally, Carlitos provides graphical representations in a table.

In conclusion, the analysis of these articles extends our knowledge about the dissemination of geometry and its teaching in an informal context aimed specifically at children in Spain at the time.

Michael P. Saclolo

Spreading the Word about Teaching with Primary Historical Sources: The TRIUMPHS Society and its Annals.

TRIUMPHS stands for TRansforming Instruction: Understanding Mathematics via Primary Historical Sources. The TRIUMPHS Society (https://triumphssociety.org/) was formed in late fall of 2022 by a group of individuals united in their collective vast experience in the use of primary historical sources in teaching mathematics and their credence in its effectiveness as a means to support student learning. Building on the legacy of three grants from the United States National Science Foundation, which supported the development and classroom testing of Primary Source Projects (PSPs), the Society's purpose is to bring together practitioners and others interested in the use of primary historical sources in mathematical teaching and learning in order to encourage and support the development and use of additional classroom resources based on primary historical sources and to publicize research based on their implementation. As its membership grows, the Society will also promote the proliferation of primary source-based pedagogy in mathematics through conversation and professional development. Through its activities, the Society hopes to encourage current and future mathematics educators to explore the use of primary historical sources in teaching and educational researchers to further examine their efficacy as a learning support.

As a means to promote the use of primary historical sources in mathematics teaching and to encourage the development of associated teaching and learning tools and research, the Society conceived of a publication called the *Annals of the TRIUMPHS Society*. Set to launch in the fall of 2024, the *Annals* will publish classroom-ready PSPs; artifacts related to such projects; and scholarly articles that support the implementation of and research on such projects and the use of primary historical sources in mathematics teaching more generally. Submissions considered for publication in the *Annals* shall undergo a robust peer review process by expert scholars and practitioners in the field. The journal will be fully open-access to potential readers and there will be no cost borne by authors in publishing their work.